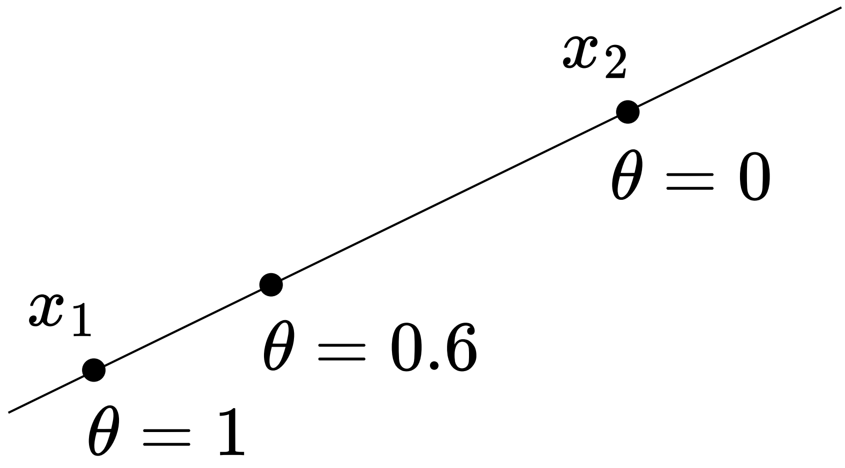
Line

Suppose x_1, x_2 are two points in $\mathbb{R}^n$. Then the line passing through them is defined as follows:

$$\boxed{x = \theta x_1 + (1-\theta)x_2, \theta \in \mathbb{R}}$$



Affine set

The set A is called **affine** if for any x_1, x_2 from A the line passing through them also lies in A, i.e.

$$\boxed{\forall \theta \in \mathbb{R}, \forall x_1, x_2 \in A : \theta x_1 + (1-\theta)x_2 \in A}$$

> 🧑‍🏫 **EXAMPLE**
> $\mathbb{R}^n$ is an affine set. The set of solutions $\{x \mid \mathbf{A}x = \mathbf{b}\}$ is also an affine set.

Related definitions

Affine combination

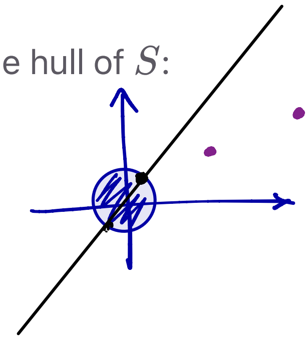
ТОЧКА
$\subset \mathbb{R}^n$

Let we have $x_1, x_2, \ldots, x_k \in S$, then the point $\boxed{\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k}$ is called affine combination of $x_1, x_2, \ldots, x_k$ if $\sum\limits_{i=1}^{k} \theta_i = 1$.

Affine hull

Множество

The set of all affine combinations of points in set S is called the affine hull of S:



$$
\mathbf{aff}(S) = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in S, \sum_{i=1}^{k} \theta_i = 1 \right\}
$$

- The set $\mathbf{aff}(S)$ is the smallest affine set containing S.

Certainly, let's translate the last two subchapters and then provide an example for the affine set definition as you requested:

Interior

The interior of the set S is defined as the following set:

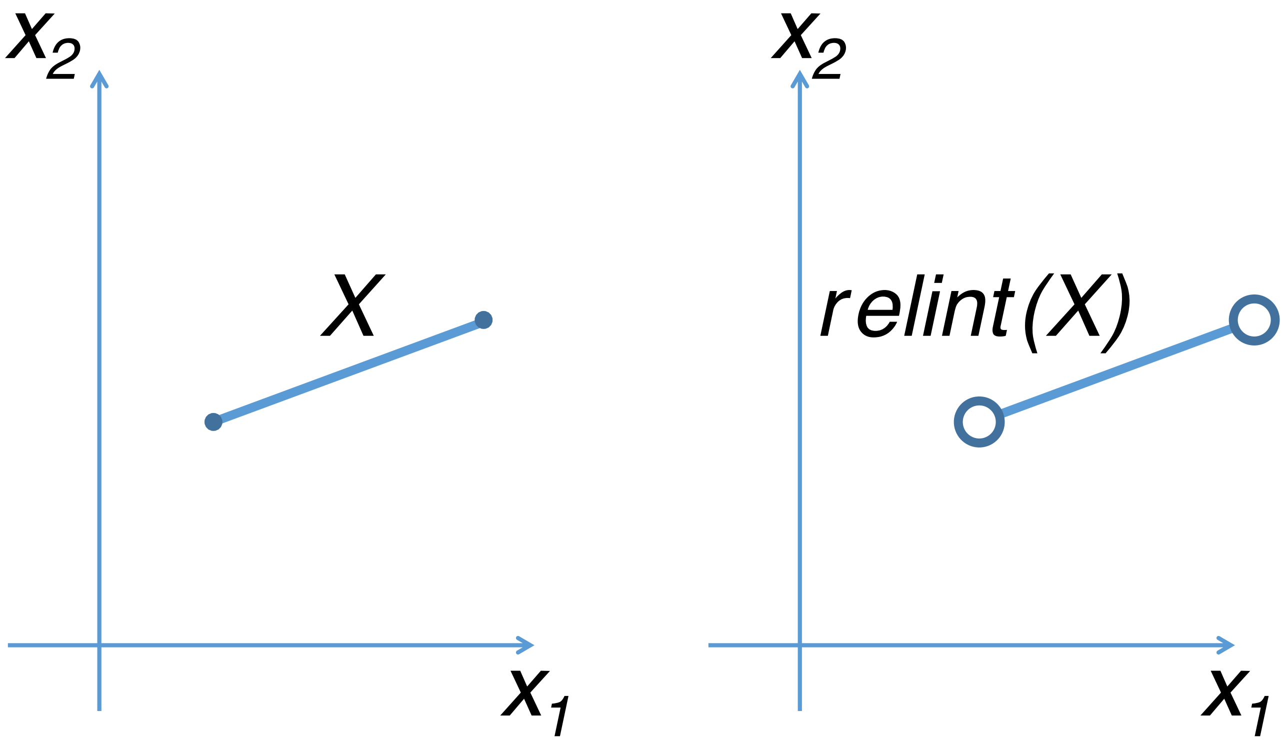
$$
\mathbf{int}(S) = \{\mathbf{x} \in S \mid \exists \varepsilon > 0,\ B(\mathbf{x}, \varepsilon) \subset S\}
$$

where $B(\mathbf{x}, \varepsilon) = \mathbf{x} + \varepsilon B$ is the ball centered at point $\mathbf{x}$ with radius ε.

Relative Interior

The relative interior of the set S is defined as the following set:

$$
\mathbf{relint}(S) = \{\mathbf{x} \in S \mid \exists \varepsilon > 0,\ B(\mathbf{x}, \varepsilon) \cap \mathbf{aff}(S) \subseteq S\}
$$



🧑‍🏫 **EXAMPLE**

Any non-empty convex set $S \subseteq \mathbb{R}^n$ has a non-empty relative interior $\mathbf{relint}(S)$.

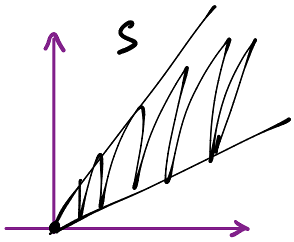
🤔 **QUESTION**

Give any example of a set $S \subseteq \mathbb{R}^n$, which has an empty interior, but at the same time has a non-empty relative interior $\mathbf{relint}(S)$.

Cone

A non-empty set S is called cone, if:

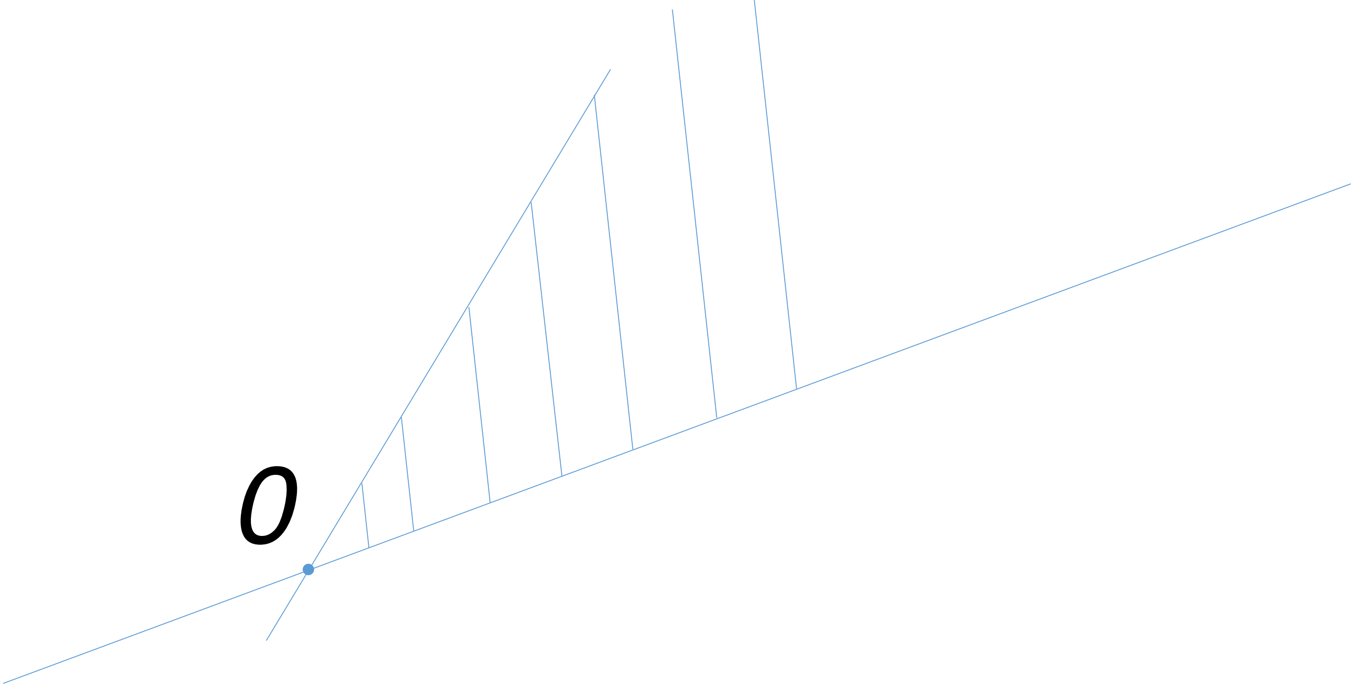
$$
\forall x \in S, \; \theta \ge 0 \quad \to \quad \theta x \in S
$$



Convex cone

The set S is called convex cone, if:

$$
\forall x_1, x_2 \in S, \; \theta_1, \theta_2 \ge 0 \quad \to \quad \theta_1 x_1 + \theta_2 x_2 \in S
$$



EXAMPLE

- $\mathbb{R}^n$
- Affine sets, containing 0
- Ray
- $\mathbf{S}^n_+$ - the set of symmetric positive semi-definite matrices

Related definitions

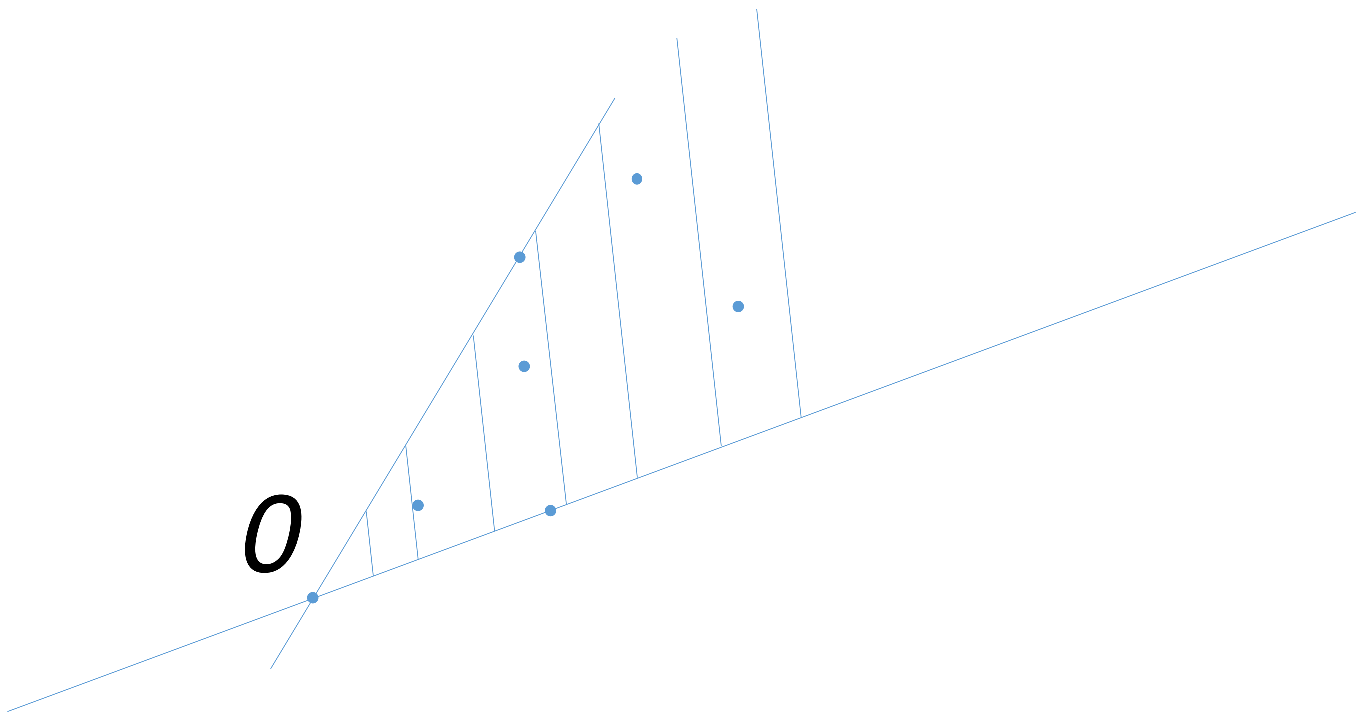
Conic combination

Let we have $x_1, x_2, \dots, x_k \in S$, then the point $\boxed{\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k}$ is called conic combination of $x_1, x_2, \dots, x_k$ if $\theta_i \geq 0$.

Conic hull

The set of all conic combinations of points in set S is called the conic hull of S:

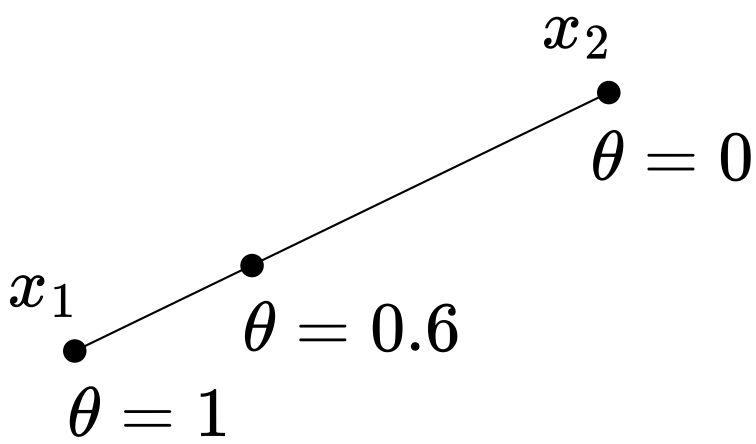
$$\mathbf{cone}(S) = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in S, \ \theta_i \geq 0 \right\}$$



Line segment

Suppose x_1, x_2 are two points in $\mathbb{R}^n$. Then the line segment between them is defined as follows:

$$x = \theta x_1 + (1-\theta)x_2, \; \theta \in [0,1]$$



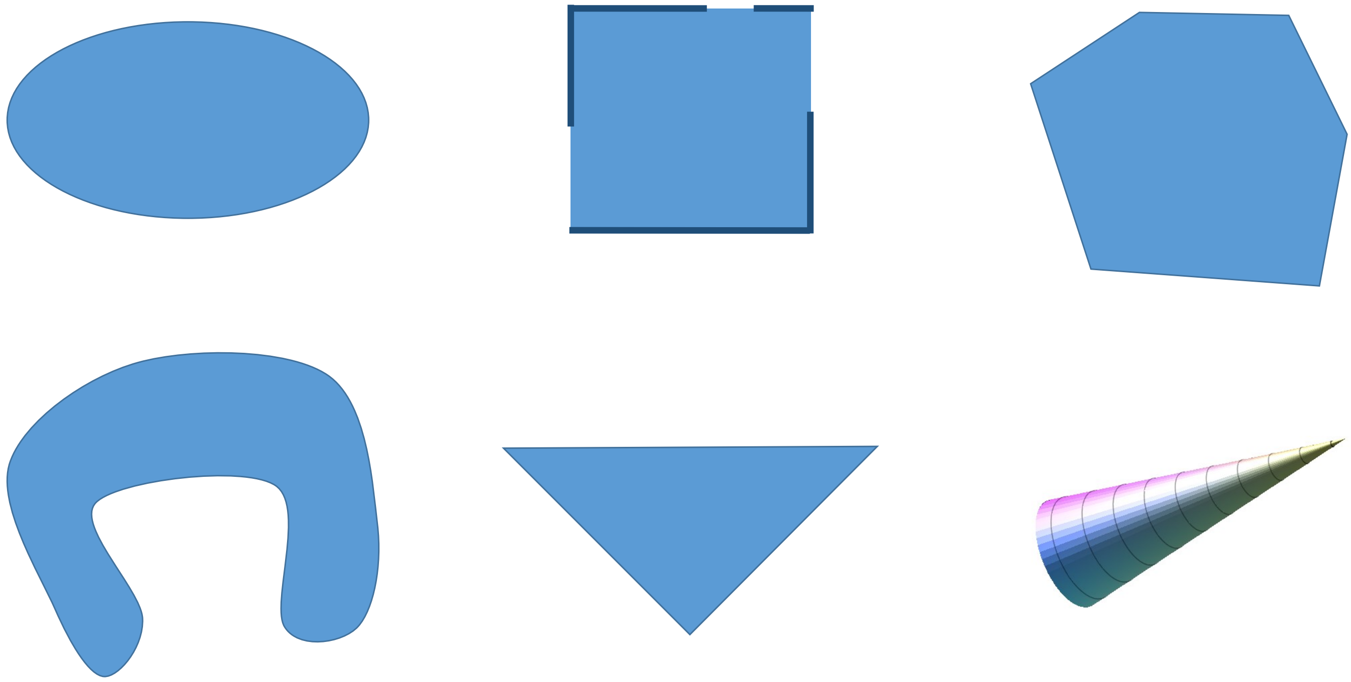
Convex set

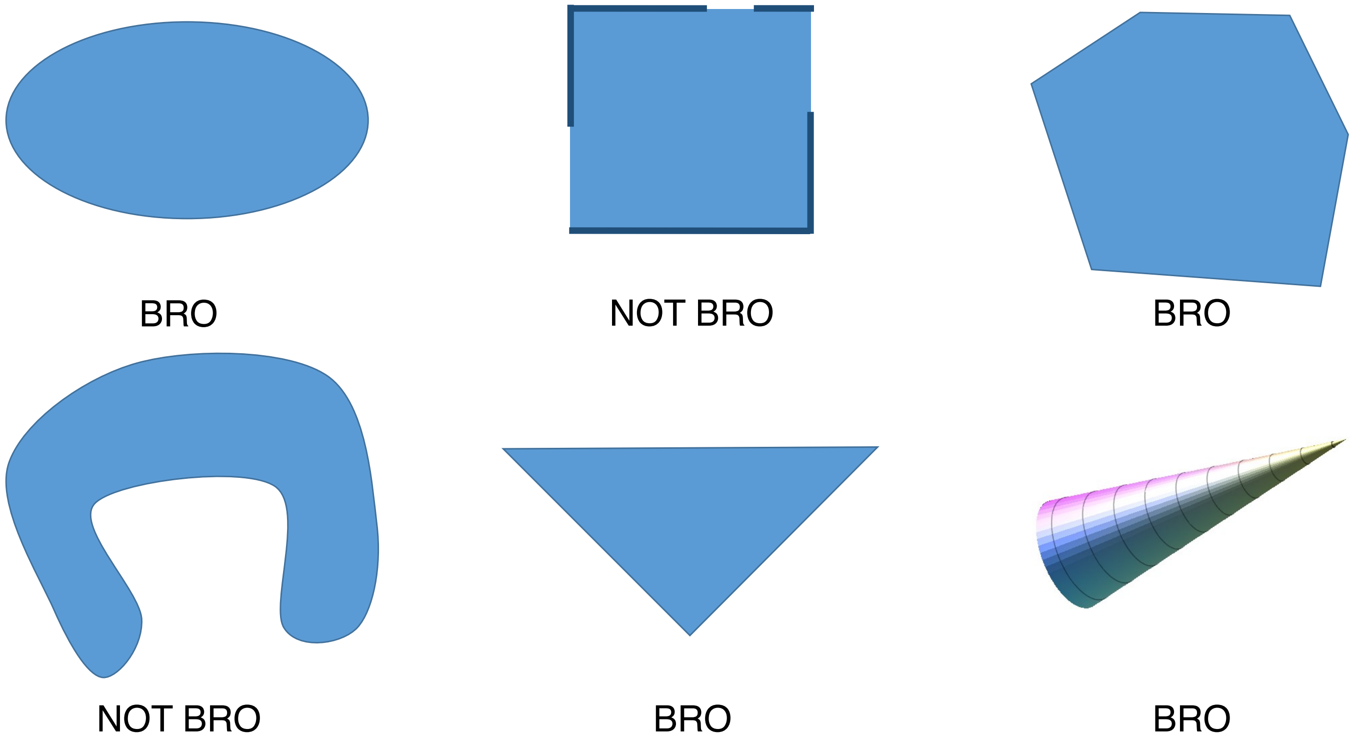
The set S is called **convex** if for any x_1, x_2 from S the line segment between them also lies in S, i.e.

$$\boxed{\begin{gathered}\forall \theta \in [0,1], \; \forall x_1, x_2 \in S: \\ \theta x_1 + (1-\theta)x_2 \in S\end{gathered}}$$

> 🧑‍🏫 **EXAMPLE**
> Empty set and a set from a single vector are convex by definition.

> 🧑‍🏫 **EXAMPLE**
> Any affine set, a ray, a line segment - they all are convex sets.





Related definitions

$x_1, x_2, \dots, x_k$

$x = \sum_{i=1}^{k} \theta_i x_i$

$\sum_i \theta_i = 1$ - Affine

$\theta_i \geq 0$ - Cone

$\sum_i \theta_i = 1$, $\theta_i \geq 0$ - convex

Convex combination

Let $x_1, x_2, \ldots, x_k \in S$, then the point $\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$ is called the convex combination of points $x_1, x_2, \ldots, x_k$ if $\sum\limits_{i=1}^k \theta_i = 1, \; \theta_i \geq 0$.

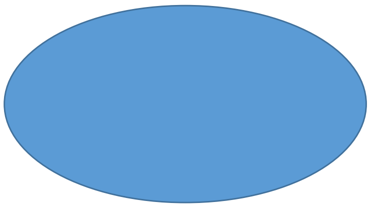
Convex hull

The set of all convex combinations of points from S is called the convex hull of the set S.

$$
\mathbf{conv}(S) = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in S, \sum_{i=1}^{k} \theta_i = 1, \; \theta_i \geq 0 \right\}
$$

- The set $\mathbf{conv}(S)$ is the smallest convex set containing S.
- The set S is convex if and only if $S = \mathbf{conv}(S)$.

Examples:



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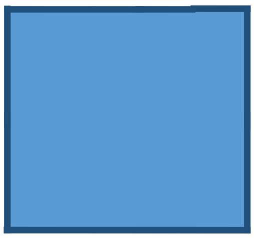

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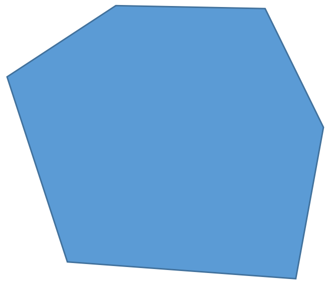

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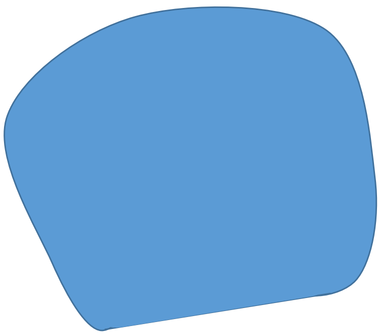

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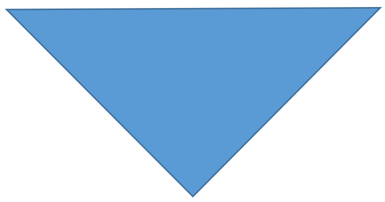

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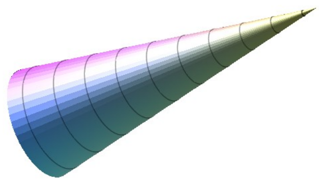


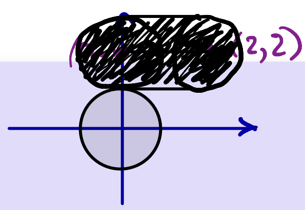
BRO

Minkowski addition

The Minkowski sum of two sets of vectors S_1 and S_2 in Euclidean space is formed by adding each vector in S_1 to each vector in S_2:

$$
\boxed{S_1 + S_2 = \{\mathbf{s_1} + \mathbf{s_2} \mid \mathbf{s_1} \in S_1, \; \mathbf{s_2} \in S_2\}}
$$

пример 1:



Similarly, one can define linear combination of the sets.

🧑‍🏫 **EXAMPLE**

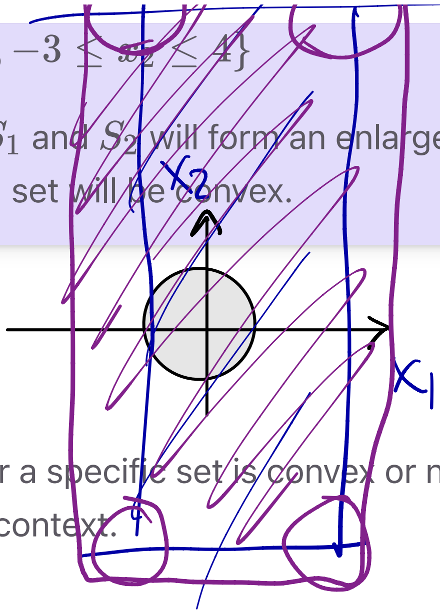
We will work in the $\mathbb{R}^2$ space. Let's define:

$$
S_1 := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}
$$

This is a unit circle centered at the origin. And:

$$S_2 := \{x \in \mathbb{R}^2 : -1 \leq x_1 \leq 2, -3 \leq x_2 \leq 4\}$$

This represents a rectangle. The sum of the sets S_1 and S_2 will form an enlarged rectangle S_2 with rounded corners. The >resulting set will be convex.



Finding convexity

In practice it is very important to understand whether a specific set is convex or not. Two approaches are used for this depending on the context.

- By definition.
- Show that S is derived from simple convex sets using operations that preserve convexity.

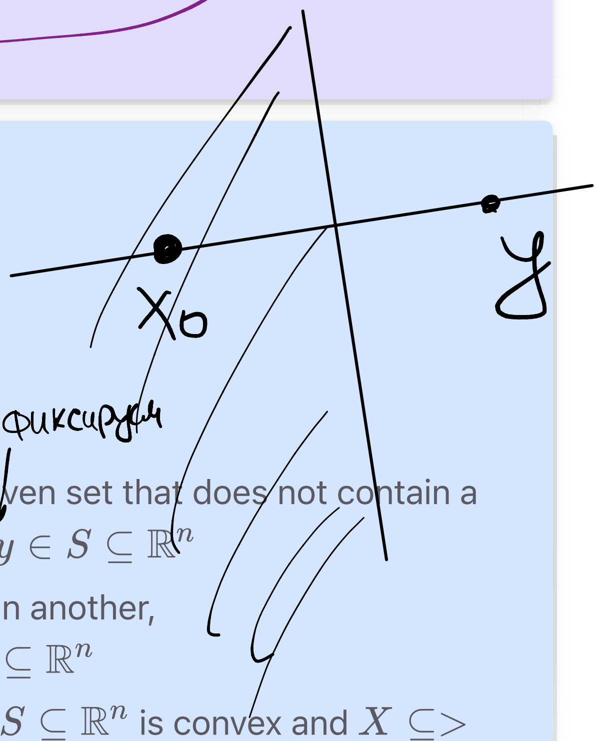
By definition

$$x_1, x_2 \in S, \; 0 \leq \theta \leq 1 \quad \rightarrow \quad \theta x_1 + (1-\theta)x_2 \in S$$

🧑‍🏫 **EXAMPLE**

Prove, that ball in $\mathbb{R}^n$ (i.e. the following set $\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\| \leq r$) - is convex.

▶ Solution



🤔 **QUESTION**

Which of the sets are convex:

- Stripe, $x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta$
- Rectangle, $x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = \overline{1,n}$
- Kleen, $x \in \mathbb{R}^n \mid a_1^\top x \leq b_1, a_2^\top x \leq b_2$
- A set of points closer to a given point than a given set that does not contain a point, $x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x-y\|_2, \forall y \in S \subseteq \mathbb{R}^n$
- A set of points, which are closer to one set than another, $x \in \mathbb{R}^n \mid \mathbf{dist}(x,S) \leq \mathbf{dist}(x,T), S, > T \subseteq \mathbb{R}^n$
- A set of points, $x \in \mathbb{R}^n \mid x + X \subseteq S$, where $S \subseteq \mathbb{R}^n$ is convex and $X \subseteq> \mathbb{R}^n$ is arbitrary.
- A set of points whose distance to a given point does not exceed a certain part of the distance to another given point is

Пример: $\{\|x - x_c\| \leq R\}$ S - выпукло?

Решение:

1) Возьмём $x_1 \in S$, $x_2 \in S$: $\|x_1 - x_c\| \leq R$

$\|x_2 - x_c\| \leq R$

2) Построим $x = \theta x_1 + (1-\theta) x_2$ $\forall \theta \in [0,1]$

$$\|x - x_c\| \leq R$$

$$\|\theta x_1 + (1-\theta)x_2 - x_c\| \leq R$$

$$\|\theta x_1 + (1-\theta)x_2 - \theta x_c - (1-\theta)x_c\| \leq R$$

$$\|\theta(x_1 - x_c) + (1-\theta)(x_2 - x_c)\| \leq R$$

$$\theta \cdot \underbrace{\|x_1 - x_c\|}_{\leq R} + (1-\theta)\underbrace{\|x_2 - x_c\|}_{\leq R} \leq R$$

$$\theta R + (1-\theta) R \leq R$$

$$R \leq R$$ ч.т.д.

$S_+^n = \{ A \in S^n : \forall x \in \mathbb{R}^n \to x^T A x \geq 0 \}$

выпукло?

$\forall x \in \mathbb{R}^n$

1) $S_1 \in S_+^n$ $\quad x^T S_1 x \geq 0$

$S_2 \in S_+^n$ $\quad x^T S_2 x \geq 0$

2) $S_\theta = \theta S_1 + (1-\theta) S_2$

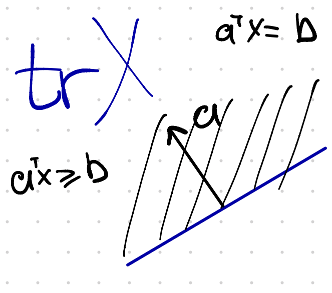
$$z^T S_\theta z = \theta \cdot z^T S_1 z + (1-\theta) z^T S_2 z \geq 0$$

Пример:

$S : \{ \tilde{S} \in S \mid tr(\tilde{S}) \geq 100500 \}$

выпукло?

$\langle X, I \rangle = tr X$



$x \in \mathbb{R}^n \mid \|x - a\|_2 \leq \theta \|xb\|_2, a, b \in \mathbb{R}^n, 0 \leq 1$

Preserving convexity

The linear combination of convex sets is convex

Let there be 2 convex sets S_x, S_y, let the set

$$S = \{s \mid s = c_1 x + c_2 y,\ x \in S_x,\ y \in S_y,\ c_1, c_2 \in \mathbb{R}\}$$

Take two points from S: $s_1 = c_1 x_1 + c_2 y_1, s_2 = c_1 x_2 + c_2 y_2$ and prove that the segment between them $\theta s_1 + (1-\theta)s_2, \theta \in [0,1]$ also belongs to S

$$\theta s_1 + (1-\theta)s_2$$

$$\theta(c_1 x_1 + c_2 y_1) + (1-\theta)(c_1 x_2 + c_2 y_2)$$

$$c_1(\theta x_1 + (1-\theta)x_2) + c_2(\theta y_1 + (1-\theta)y_2)$$

$$c_1 x + c_2 y \in S$$

The intersection of any (!) number of convex sets is convex

If the desired intersection is empty or contains one point, the property is proved by definition. Otherwise, take 2 points and a segment between them. These points must lie in all intersecting sets, and since they are all convex, the segment between them lies in all sets and, therefore, in their intersection.

The image of the convex set under affine mapping is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \quad \rightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex} \qquad (f(x) = \mathbf{A}x + \mathbf{b})$$

Examples of affine functions: extension, projection, transposition, set of solutions of linear matrix inequality $\{x \mid x_1 A_1 + \ldots + x_m A_m \preceq B\}$. Here $A_i, B \in \mathbf{S}^p$ are symmetric matrices $p \times p$.

Note also that the prototype of the convex set under affine mapping is also convex.

$$S \subseteq \mathbb{R}^m \text{ convex} \ \rightarrow \ f^{-1}(S) = \{x \in \mathbb{R}^n \mid f(x) \in S\} \text{ convex} \ \ (f(x) = \mathbf{A}x + \mathbf{b})$$

🧑‍🏫 **EXAMPLE**

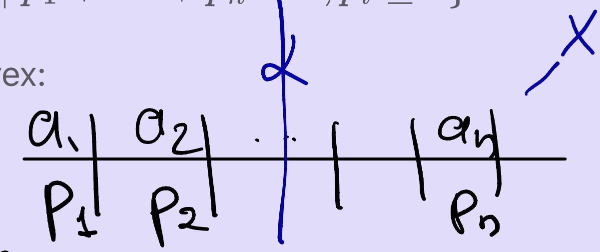
Let $x \in \mathbb{R}$ is a random variable with a given probability distribution of $\mathbb{P}(x = a_i) = p_i$, where $i = 1, > \ldots, n$, and $a_1 < \ldots < a_n$. It is said that the probability vector of outcomes of $p \in \mathbb{R}^n$ belongs to the >probabilistic simplex, i.e.

$$P = \{p \mid \mathbf{1}^T p = 1, p \succeq 0\} = \{p \mid p_1 + \ldots + p_n = 1, p_i \geq 0\}.$$

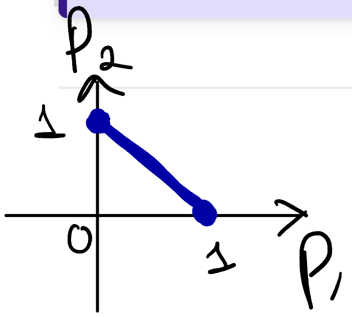
Determine if the following sets of p are convex:

- $\mathbb{P}(x > \alpha) \leq \beta$
- $\mathbb{E}|x^{201}| \leq \alpha \mathbb{E}|x|$
- $\mathbb{E}|x^2| \geq \alpha \mathbb{V}x \geq \alpha$

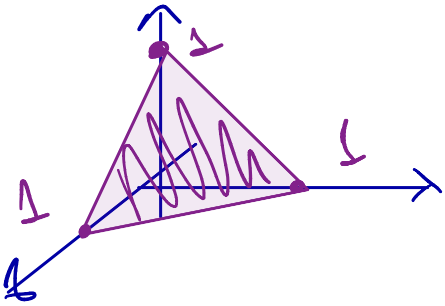
▶ Solution



$$\Delta^n = \left\{ p \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n p_i = 1, \; p_i \geq 0 \right\}$$



$$\boxed{a.\ \mathbb{E}x \leq 100500 \quad \text{выпукло}}$$



$$\sum_{i=1}^n p_i \cdot a_i \leq 100500$$

выпукло

$$\mathbb{P}(x > \alpha) \leq \beta$$

$$\sum_{i=j} p_i < \beta \qquad c^T p \leq \beta$$

$$c = \begin{array}{c c} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ j \rightarrow & \end{array}$$

$$\mathbb{E}|x^{201}| = \sum_{i=1}^n p_i \cdot |a_i^{201}| = d^T p$$

$$\mathbb{V}x = \mathbb{E}x^2 - (\mathbb{E}x)^2 = \sum a_i^2 p_i - \left(\sum_i a_i p_i\right)^2$$