Conjugate (dual) function

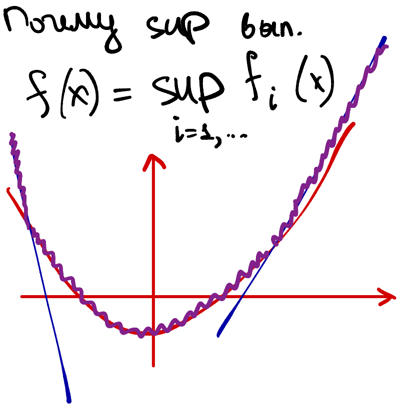
супремум
инфимум !

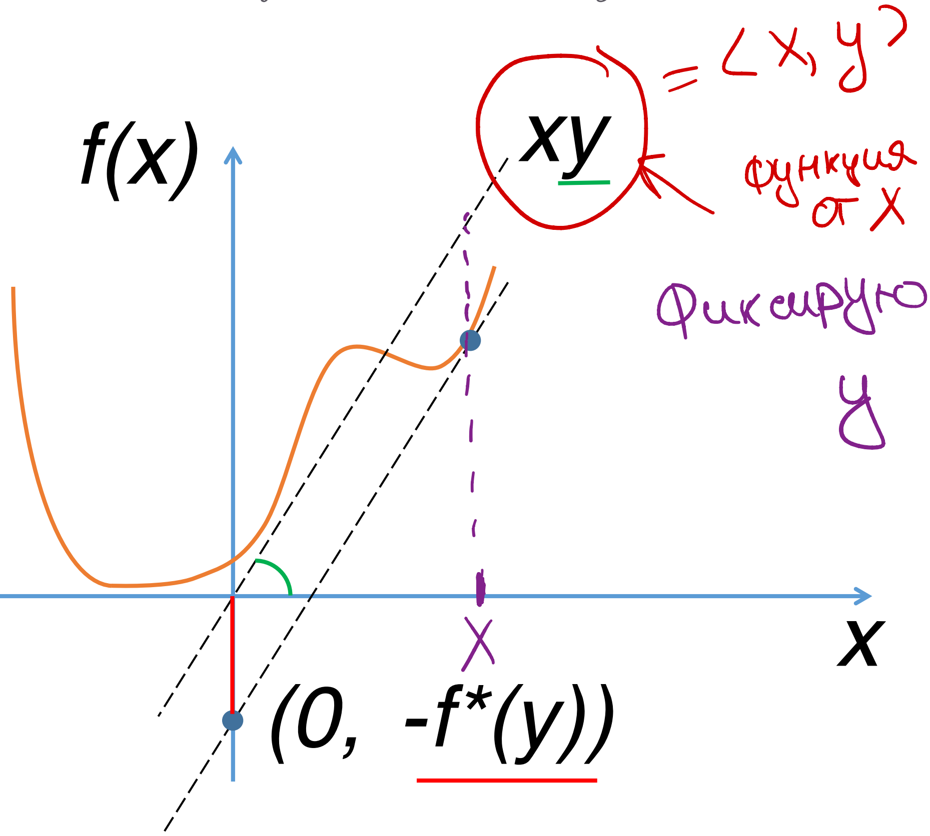
Let $f : \mathbb{R}^n \to \mathbb{R}$. The function $f^* : \mathbb{R}^n \to \mathbb{R}$ is called convex conjugate (Fenchel's conjugate, dual, Legendre transform) $f(x)$ and is defined as follows:

преобразование Фенхеля-Лежандра
Фенхеля

$$
\boxed{f^*(y) = \sup_{x \in \mathbf{dom}\, f} (\langle y, x\rangle - f(x)).} = \sup_{x \in dom f} \varphi(x,y)
$$

Let's notice, that the domain of the function f^* is the set of those y, where the supremum is finite.





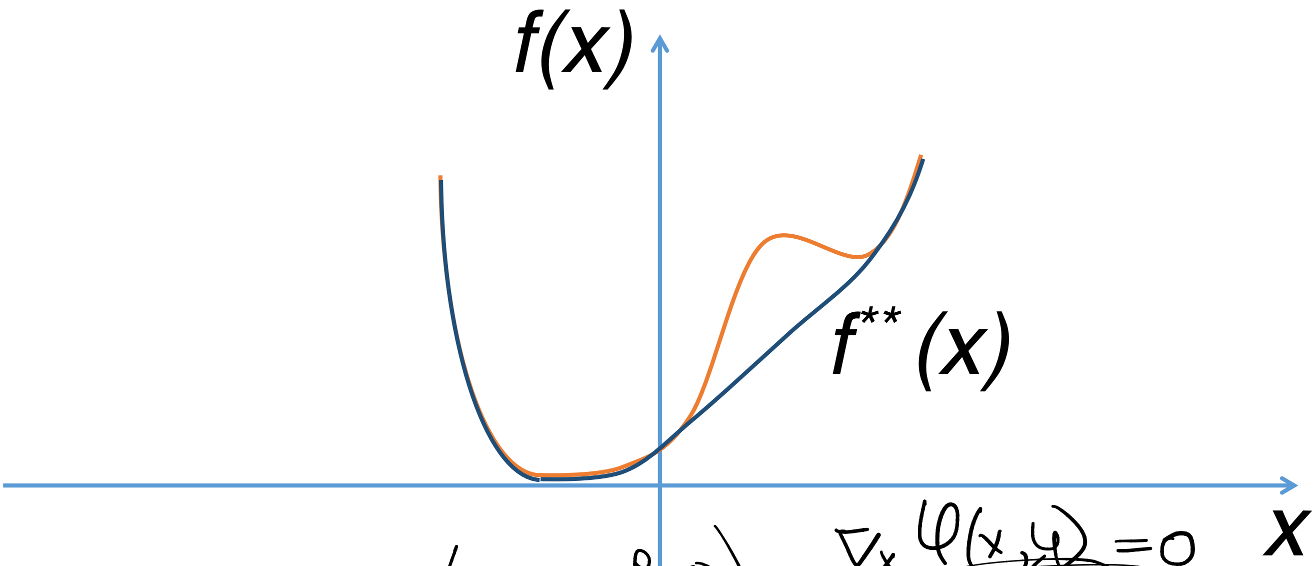
Properties

- $\boxed{f^*(y) \text{ - is always a closed convex function}}$ (a point-wise supremum of closed convex functions) on y. (Function $f : X \to R$ is called closed if $\mathbf{epi}(f)$ is a closed set in $X \times R$.)
- Fenchel–Young inequality:

$$
\boxed{f(x) + f^*(y) \geq \langle y, x \rangle}
$$

- Let the functions $f(x), f^*(y), f^{**}(x)$ be defined on the $\mathbb{R}^n$. Then $f^{**}(x) = f(x)$ if and only if $f(x)$ - is a proper convex function (Fenchel - Moreau theorem). (proper convex function = closed convex function)

- Consequence from Fenchel–Young inequality: $f(x) \geq f^{**}(x)$.



$\nabla_x \varphi(x, y) = 0$

$f^*(y) = \sup\limits_{x \in \mathrm{dom} f} \left(\langle x, y \rangle - f(x) \right)$

$y = \nabla f(x^*)$

- In case of differentiable function, $f(x)$ - convex and differentiable, $\mathbf{dom}\ f = \mathbb{R}^n$. Then $x^\star = \underset{x}{\operatorname{argmin}} \langle x, y \rangle - f(x)$. Therefore $y = \nabla f(x^\star)$. That's why:

$$\boxed{f^\star(y) = \langle \nabla f(x^\star), x^\star \rangle - f(x^\star)} = \varphi(x^*, y)$$

$$\boxed{f^\star(y) = \langle \nabla f(z), z \rangle - f(z),} \qquad y = \nabla f(z), \quad z \in \mathbb{R}^n$$

- Let $\boxed{f(x, y) = f_1(x) + f_2(y),}$ where f_1, f_2 - convex functions, then

$$\boxed{f^*(p, q) = f_1^*(p) + f_2^*(q)}$$

- Let $\boxed{f(x) \leq g(x)}$ $\forall x \in X$. Let also $f^\star(y), g^\star(y)$ be defined on Y. Then $\forall x \in X, \forall y \in Y$

$$f^\star(y) \geq g^\star(y) \qquad f^{\star\star}(x) \leq g^{\star\star}(x)$$

Examples

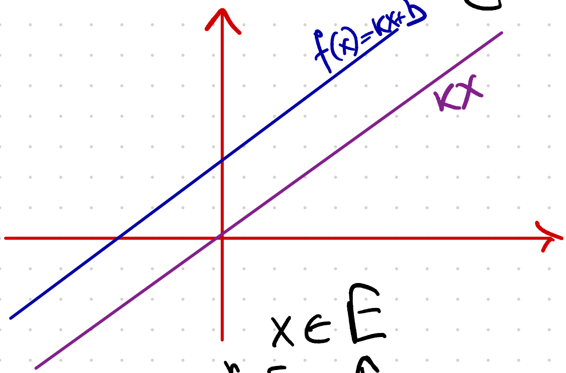
The scheme of recovering the convex conjugate is pretty algorithmic:

1. Write down the definition $f^\star(y) = \sup\limits_{x \in \mathbf{dom}\ f} \left(\langle y, x \rangle - f(x) \right) = \sup\limits_{x \in \mathbf{dom}\ g} g(x, y)$.
2. Find those y, where $\sup\limits_{x \in \mathbf{dom}\ g} g(x, y)$ is finite. That's the domain of the dual function $f^\star(y)$.
3. Find $x^\star$, which maximize $g(x, y)$ as a function on x. $f^\star(y) = g(x^\star, y)$.

Пример 1:

$f(x) = kx + b$, $x \in \mathbb{R}$

$f^*(y) = ?$ $\quad f^*(y) = \sup\limits_x (xy - f(x))$



$x \in E$ ↑ прямое пр-во

$R = \mathbb{R}^n = E$

Область определения $f^*(y)$: такие y ($\in E^*$ ↑ сопр. пр-во, пр-во всех лин. функц над ним) для которых $\sup\limits_x \varphi(x,y)$ конечен ($< \infty$)

Boyd

Решение:

1) Если $y \neq k$, то $\sup \varphi(x,y)$ бесконечен

$\Rightarrow \operatorname{dom} f^* = \{k\}$

$$f^*(k) = kx - (kx + b) = -b$$

Ответ: $f^*(y) = -b$, $y = k$.

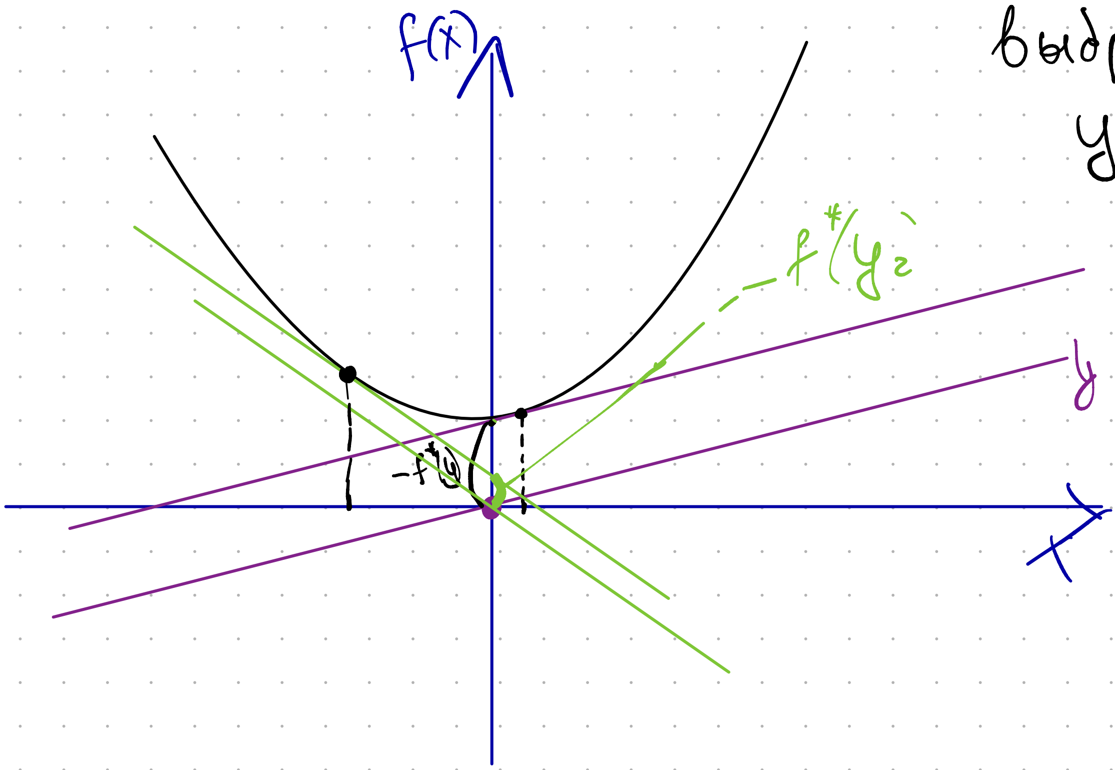
упр: Найти f^*, если

$$f(x) = -b, \quad x = K.$$

Опр.: $f^*(y) = \sup\limits_{x \in \{K\}} \left[x \cdot y - f(x) \right] =$

$= y \cdot K - (-b) = \boxed{Ky + b}$

$$\boxed{f^{**}(x) = f(x)}$$



выбрали
y, получили
смещение
касательной

EXAMPLE

Find $f^*(y)$, if $f(x) = ax + b$.

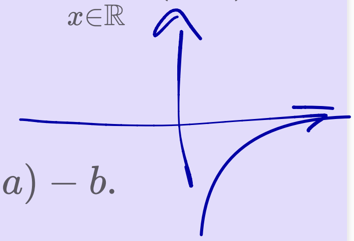
▼ Solution

1. By definition:

$f \to \min\limits_{x,y,z}$

$$f^*(y) = \sup_{x \in \mathbb{R}} [yx - f(x)] = \sup_{x \in \mathbb{R}} g(x,y) \quad \textbf{dom } f^* = \{y \in \mathbb{R} : \sup_{x \in \mathbb{R}} g(x,y) \text{ is fi}$$

2. Consider the function whose supremum is the conjugate:



$$g(x,y) = yx - f(x) = yx - ax - b = x(y-a) - b.$$

3. Let's determine the domain of the function (i.e. those y for which sup is finite). This is a single point, $y = a$. Otherwise one may choose such x

4. Thus, we have: $\textbf{dom } f^* = \{a\}$; $f^*(a) = -b$

$\nabla_x \varphi(x,y) = 0$

$f^*(y) = \sup\limits_{x \in \mathbb{R}_{++}} (xy - f(x)) = $

$y - (-1)x^{-2} = 0$

QUESTION

Find $f^*(y)$, if $f(x) = \dfrac{1}{x}, \quad x \in \mathbb{R}_{++}$.

$= \sup\limits_{x \in \mathbb{R}_{++}} \left(xy - \dfrac{1}{x}\right)$

$x^{-2} = -y \qquad \tilde{x} = \dfrac{1}{\sqrt{-y}}$

$f^*(y) = \varphi(\tilde{x}, y) = \dfrac{1}{\sqrt{-y}} \cdot y - \dfrac{1}{1/\sqrt{-y}} = -\sqrt{-y} - \sqrt{-y} = -2\sqrt{-y}$

EXAMPLE

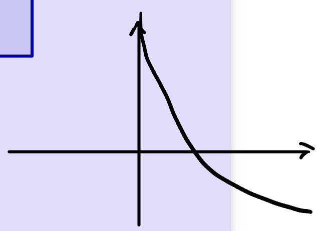
Find $\boxed{f^*(y), \text{ if } f(x) = -\log x,} \quad x \in \mathbb{R}_{++}$.

Ответ: $f^*(y) = -2\sqrt{-y}$
$y \leq 0$

▼ Solution

1. Consider the function whose supremum defines the conjugate:

$$\boxed{g(x,y) = \langle y, x\rangle - f(x)} = yx + \log x.$$



2. This function is unbounded above when $y \geq 0$. Therefore, the domain of f^* is $\textbf{dom } f^* = \{y < 0\}$.

3. This function is concave and its maximum is achieved at the point with zero gradient:

$$\boxed{\frac{\partial}{\partial x}(yx + \log x) = \frac{1}{x} + y = 0.}$$

$x = -\dfrac{1}{y}$

Thus, we have $x = -\dfrac{1}{y}$ and the conjugate function is:

$$\boxed{f^*(y) = -\log(-y) - 1.}$$

$y < 0$

EXAMPLE

Find $f^*(y)$, if $f(x) = e^x$.

▼ Solution

1. Consider the function whose supremum defines the conjugate:

$$\boxed{g(x,y) = \langle y, x\rangle - f(x) = yx - e^x.}$$

2. This function is unbounded above when $y < 0$. Thus, the domain of f^* is $\mathbf{dom}\ f^* = \{y \geq 0\}$.
3. The maximum of this function is achieved when $\boxed{x = \log y}$. Hence:

$$\boxed{f^*(y) = y \log y - y,}$$

assuming $0 \log 0 = 0$.

$(t+1)\log(t+1) - t - 1$

$\frac{y}{e} = t \Rightarrow f^*(t) = t \cdot e \log et - et =$

$= te(\ln t + 1) - et$

$= \ln t \cdot te$

EXAMPLE

Find $f^*(y)$, if $\boxed{f(x) = x \log x}$, $x \neq 0$, and $f(0) = 0$, $\quad x \in \mathbb{R}_+$.

▼ Solution

1. Consider the function whose supremum defines the conjugate:

$$g(x,y) = \langle y, x\rangle - f(x) = xy - x\log x.$$

2. This function is upper bounded for all y. Therefore, $\mathbf{dom}\ f^* = \mathbb{R}$.
3. The maximum of this function is achieved when $x = e^{y-1}$. Hence:

$$f^*(y) = e^{y-1}.$$

$y - 1 = t$

EXAMPLE

Find $f^*(y)$, if $f(x) = \frac{1}{2}x^TAx, \quad A \in \mathbb{S}^n_{++}$.

▼ Solution

1. Consider the function whose supremum defines the conjugate:

$$g(x,y) = \langle y, x\rangle - f(x) = y^Tx - \frac{1}{2}x^TAx.$$

2. This function is upper bounded for all y. Thus, $\mathbf{dom}\ f^* = \mathbb{R}$.
3. The maximum of this function is achieved when $x = A^{-1}y$. Hence:

$$f^*(y) = \frac{1}{2} y^T A^{-1} y.$$

🧑‍🏫 **EXAMPLE**

Find $f^*(y)$, if $f(x) = \max\limits_i x_i, \quad x \in \mathbb{R}^n$.

▼ Solution

1. Consider the function whose supremum defines the conjugate:

$$g(x, y) = \langle y, x \rangle - f(x) = y^T x - \max_i x_i.$$

2. Observe that if vector y has at least one negative component, this function is not bounded by x.
3. If $y \succeq 0$ and $1^T y > 1$, then $y \notin \textbf{dom } f^*(y)$.
4. If $y \succeq 0$ and $1^T y < 1$, then $y \notin \textbf{dom } f^*(y)$.
5. Only left with $y \succeq 0$ and $1^T y = 1$. In this case, $x^T y \le \max\limits_i x_i$.
6. Hence, $f^*(y) = 0$.

🧑‍🏫 **EXAMPLE**

Revenue and profit functions. We consider a business or enterprise that consumes n resources and produces a product that can be sold. We let $r = (r_1, \ldots, r_n)$ denote the vector of resource quantities consumed, and $S(r)$ denote the sales revenue derived from the product produced (as a function of the resources consumed). Now let p_i denote the price (per unit) of resource i, so the total amount paid for resources by the enterprise is $p^\top r$. The profit derived by the firm is then $S(r) - p^\top r$. Let us fix the prices of the resources, and ask what is the maximum profit that can be made, by wisely choosing the quantities of resources consumed. This maximum profit is given by

доход   расход   p – вектор цен

прибыль

$$\boxed{M(p) = \sup_r \left(S(r) - p^\top r \right)}$$

The function $M(p)$ gives the maximum profit attainable, as a function of the resource prices. In terms of conjugate functions, we can express M as $M(p) = (-S)^*(-p)$. Thus the maximum profit (as a function of resource prices) is closely related to the conjugate of gross sales (as a function of resources consumed).

$$S^*(p) = \sup_r \left[p^T r - S(r) \right]$$

$f \to \min\limits_{x,y,z}$

Dual norm

Let $\|x\|$ be the norm in the primal space $x \in S \subseteq \mathbb{R}^n$, then the following expression defines dual norm:

если есть $\|\cdot\|$

$$
\boxed{\|x\|_\star = \sup_{\|y\| \leq 1} \langle y, x \rangle}
$$

The intuition for the finite-dimensional space is how the linear function (element of the dual space) $f_y(\cdot)$ could stretch the elements of the primal space with respect to their size, i.e. $\|y\|_* = \sup\limits_{x \neq 0} \dfrac{\langle y, x \rangle}{\|x\|}$

$$
\boxed{\|y\|_* = \sup_{\|x\| \leq 1} \langle y, x \rangle}
$$

Properties

САГА о двух мирах

Шок-контент: $\nabla f(x) \subset E^*$

- One can easily define the dual norm as:

$$
\|x\|_* = \sup_{y \neq 0} \frac{\langle y, x \rangle}{\|y\|}
$$

- The dual norm is also a norm itself $\langle y, y \rangle$
- For any $x \in E, y \in E^*$: $\boxed{x^\top y \leq \|x\| \cdot \|y\|_*}$
- $\boxed{(\|x\|_p)_* = \|x\|_q}$ if $\boxed{\frac{1}{p} + \frac{1}{q} = 1}$, where $p, q \geq 1$

🧑‍🏫 **EXAMPLE**

The Euclidian norm is self dual $\boxed{(\|x\|_2)_\star = \|x\|_2.}$

Examples

🧑‍🏫 **EXAMPLE**

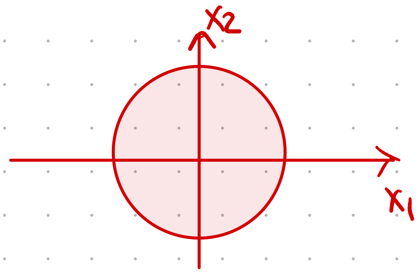
Let $\boxed{f(x) = \|x\|}$. Prove that $\boxed{f^\star(y) = \mathbb{O}_{\|y\|_* \leq 1}}$

▼ Solution

$$
f^*(y) = \sup_{x \in \mathbb{R}^n} \left(\langle x, y \rangle - \|x\| \right)
$$

$$
\langle x, y \rangle \leq \|x\| \cdot \|y\|_*
$$

Решение: $\varphi(x,y) = \langle x,y\rangle - \|x\|$

1) 

$\langle x,y\rangle \le \|x\|\cdot\|y\|_*$

$\langle x,y\rangle - \|x\| \le \|x\|\cdot\|y\|_* - \|x\|$

$\langle x,y\rangle - \|x\| \le \|x\|\left(\|y\|_* - 1\right)$

100500

-1

$\|y\|_* \le 1$

$\|y\|_* > 1$

Докажем, что если $\|y\|_* > 1$, то

dom f

$\sup\limits_x \langle x,y\rangle - \|x\|$ бесконечен,

предъявим $x = y$

$\|x\|^2 - \|x\|$

2) докажем, что $\forall y: \|y\|_* \le 1$

$\sup\limits_x \langle x,y\rangle - \|x\| =$

$= 0 \Rightarrow$ Ответ: $f^*(y) =$