# Gradient Descent

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Optimization methods. MIPT

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Thus, the direction of the antigradient

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gives the direction of the **steepest local** decreasing of the function $f$. The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

# Gradient flow ODE

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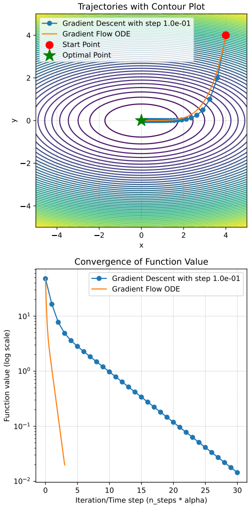

Figure 1: Gradient flow trajectory

# Necessary local minimum condition

$$
\begin{aligned}
& f'(x) = 0 \\
& -\eta f'(x) = 0 \\
& x - \eta f'(x) = x \\
& x_k - \eta f'(x_k) = x_{k+1}
\end{aligned}
$$

# Minimizer of Lipschitz parabola

If a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and its gradient satisfies Lipschitz conditions with constant $L$, then $\forall x, y \in \mathbb{R}^n$:

$$|f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2} \|y - x\|^2,$$

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which geometrically means, that if we'll fix some point $x_0 \in \mathbb{R}^n$ and define two parabolas:

$$\phi_1(x) = f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle - \frac{L}{2} \|x - x_0\|^2,$$

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Now, if we have global upper bound on the function, in a form of parabola, we can try to go directly to its minimum.

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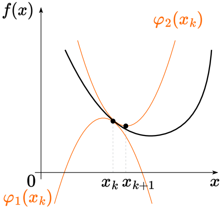

Figure 2: Illustration

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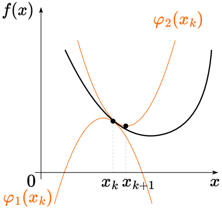

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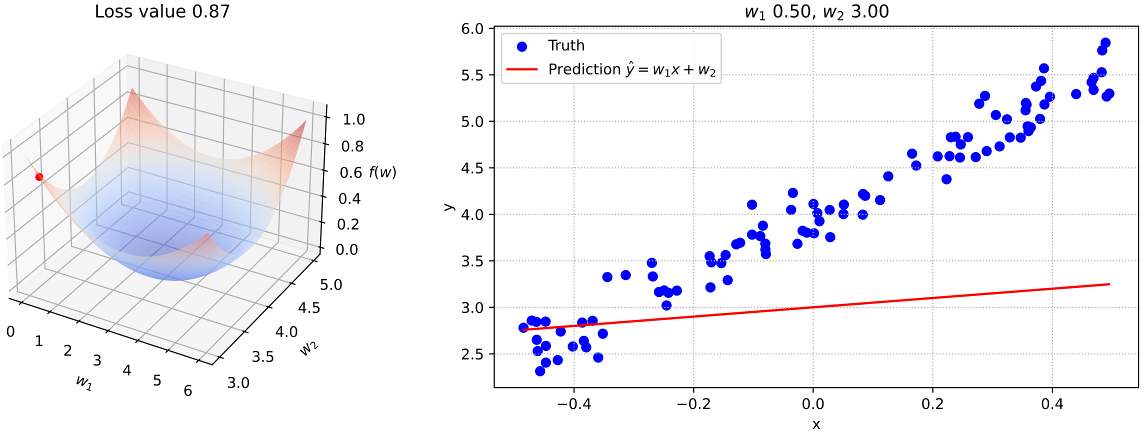
$$\nabla \phi_2(x) = 0$$
$$\nabla f(x_0) + L(x^* - x_0) = 0$$
$$x^* = x_0 - \frac{1}{L} \nabla f(x_0)$$
$$x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$$

This way leads to the $\frac{1}{L}$ stepsize choosing. However, often the $L$ constant is not known.

# Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate $\alpha$:



# Exact line search aka steepest descent

$$\alpha_k = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. Interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

$$\alpha_k = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

Optimality conditions:

$$\nabla f(x_{k+1})^\top \nabla f(x_k) = 0$$

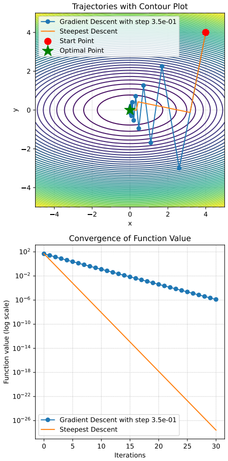


Figure 3: Steepest Descent

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# Convergence rates

$$\min_{x \in \mathbb{R}^n} f(x) \qquad\qquad x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

| smooth | convex | smooth & convex | smooth & strongly convex (or PL) |
|---|---|---|---|
| $\|\nabla f(x_k)\|^2 \approx \mathcal{O}\left(\frac{1}{k}\right)$ | $f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$ | $f(x_k) - f^* \approx \mathcal{O}\left(\frac{1}{k}\right)$ | $\|x_k - x^*\|^2 \approx \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^k\right)$ |

# Gradient Descent convergence. Smooth convex case

# Gradient Descent convergence. Smooth $\mu$-strongly convex case

# Gradient Descent convergence. Polyak-Lojasiewicz case