Gradient Descent. Non-smooth case. Linear Least squares with l_1-regularization.

Daniil Merkulov

Optimization methods. MIPT

Previously

- Gradient Descent. Convergence for strongly convex quadratic function. Optimal hyperparameters.

$$\alpha = \frac{2}{\mu + L} \quad \varkappa = \frac{L}{\mu} \geq 1 \quad \rho = \frac{\varkappa - 1}{\varkappa + 1}$$

$$\|x_k - x^*\| \leq \rho^k \|x_0 - x^*\|$$

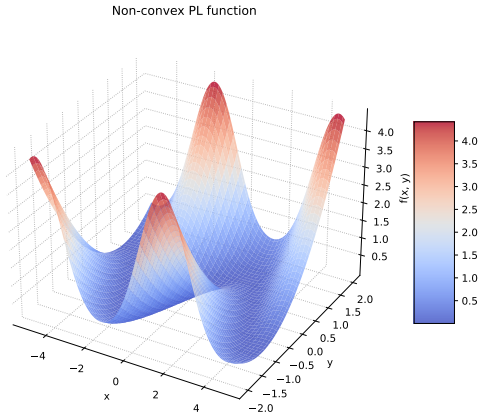

Figure 1: PL function

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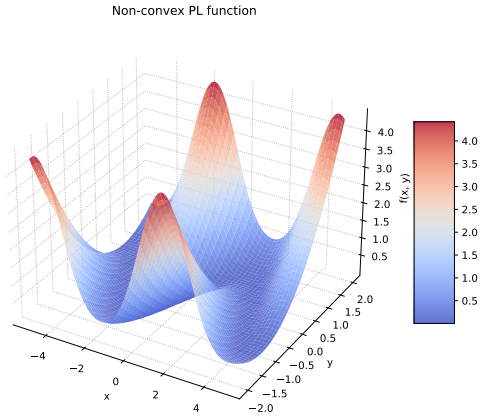

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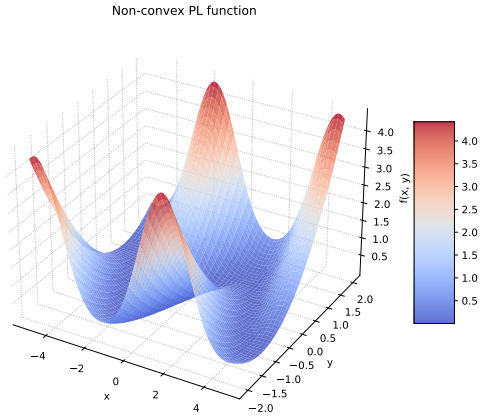

Figure 1: PL function

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL-function.

Proof

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

Putting $y = x^*$:

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Let $a = \frac{1}{\sqrt{\mu}}\nabla f(x)$ and
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Let $a = \frac{1}{\sqrt{\mu}}\nabla f(x)$ and
$b = \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)$
Then $a + b = \sqrt{\mu}(x - x^*)$ and
$a - b = \frac{2}{\sqrt{\mu}}\nabla f(x) - \sqrt{\mu}(x - x^*)$

Any μ-strongly convex differentiable function is a PL-function

$$f(x) - f(x^*) \leq \frac{1}{2}\left(\frac{1}{\mu}\|\nabla f(x)\|_2^2 - \left\|\sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)\right\|_2^2\right)$$

which is exactly PL-condition. It means, that we already have linear convergence proof for any strongly convex function.

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$$f(x) - f(x^*) \leq \frac{1}{2\mu}\|\nabla f(x)\|_2^2,$$

which is exactly PL-condition. It means, that we already have linear convergence proof for any strongly convex function.

Non-smooth optimization

$$\min_{x \in \mathbb{R}^n} f(x),$$

A classical convex optimization problem is considered. We assume that $f(x)$ is a convex function, but now we do not require smoothness.

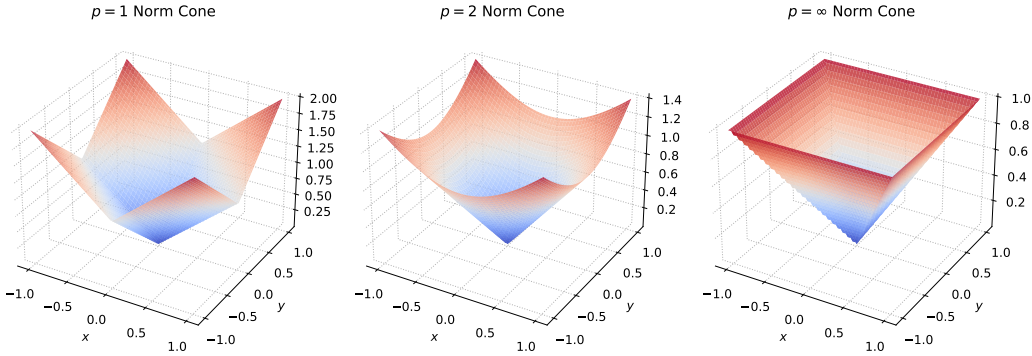

Figure 2: Norm cones for different p - norms are non-smooth

Non-smooth optimization

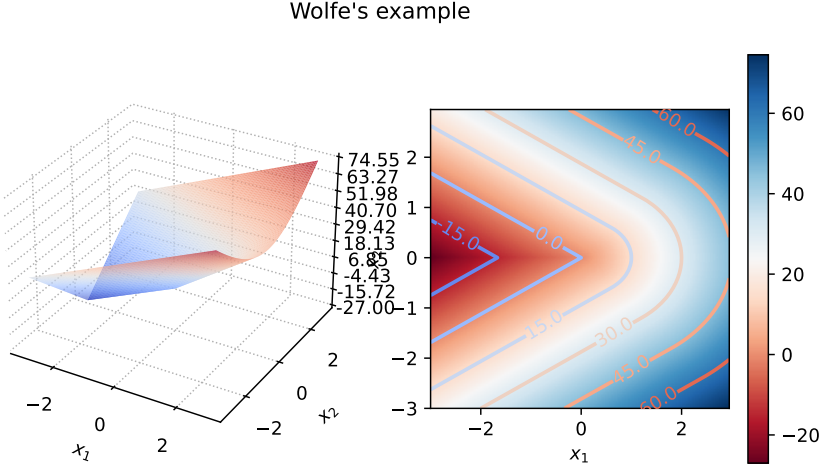

Figure 3: Wolfe's example. Open in Colab

Algorithm

A vector g is called the **subgradient** of the function $f(x) : S \to \mathbb{R}$ at the point x_0 if $\forall x \in S$:

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$

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The idea is very simple: let's replace the gradient $\nabla f(x_k)$ in the gradient descent algorithm with a subgradient g_k at point x_k:

$$x_{k+1} = x_k - \alpha_k g_k, \tag{SD}$$

where g_k is an arbitrary subgradient of the function $f(x)$ at the point x_k, $g_k \in \partial f(x_k)$

Convergence bound

$$\|x_{k+1} - x^*\|^2 = \|x_k - x^* - \alpha_k g_k\|^2 =$$

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Convergence bound

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \|x_k - x^* - \alpha_k g_k\|^2 = \\ &= \|x_k - x^*\|^2 + \alpha_k^2 \|g_k\|^2 - 2\alpha_k \langle g_k, x_k - x^* \rangle \\ 2\alpha_k \langle g_k, x_k - x^* \rangle &= \|x_k - x^*\|^2 + \alpha_k^2 \|g_k\|^2 - \|x_{k+1} - x^*\|^2\end{aligned}$$

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Convergence bound

$$\|x_{k+1} - x^*\|^2 = \|x_k - x^* - \alpha_k g_k\|^2 =$$
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- We additionaly assume, that $\|g_k\|^2 \leq G^2$
- We use the notation $R = \|x_0 - x^*\|_2$

Convergence bound

Assuming $\alpha_k = \alpha$ (constant stepsize), we have:

$$\sum_{k=0}^{T-1} \langle g_k, x_k - x^* \rangle \leq \frac{R^2}{2\alpha} + \frac{\alpha}{2} G^2 T$$

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Minimizing the right-hand side by α gives $\alpha^* = \frac{R}{G}\sqrt{\frac{1}{T}}$ and $\sum_{k=0}^{T-1} \langle g_k, x_k - x^* \rangle \leq GR\sqrt{T}$.

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$$\begin{aligned} f(\overline{x}) - f^* &= f\left(\frac{1}{T}\sum_{k=0}^{T-1} x_k\right) - f^* \leq \frac{1}{T}\left(\sum_{k=0}^{T-1}(f(x_k) - f^*)\right) \\ &\leq \frac{1}{T}\left(\sum_{k=0}^{T-1}\langle g_k, x_k - x^* \rangle\right) \\ &\leq GR\frac{1}{\sqrt{T}} \end{aligned}$$

Important notes:

- Obtaining bounds not for x_T but for the arithmetic mean over iterations $\overline{x}$ is a typical trick in obtaining estimates for methods where there is convexity but no monotonic decreasing at each iteration. There is no guarantee of success at each iteration, but there is a guarantee of success on average

Convergence bound

Assuming $\alpha_k = \alpha$ (constant stepsize), we have:

$$\sum_{k=0}^{T-1} \langle g_k, x_k - x^* \rangle \leq \frac{R^2}{2\alpha} + \frac{\alpha}{2} G^2 T$$

Minimizing the right-hand side by α gives $\alpha^* = \frac{R}{G}\sqrt{\frac{1}{T}}$ and $\sum_{k=0}^{T-1} \langle g_k, x_k - x^* \rangle \leq GR\sqrt{T}$.

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Important notes:

- Obtaining bounds not for x_T but for the arithmetic mean over iterations $\overline{x}$ is a typical trick in obtaining estimates for methods where there is convexity but no monotonic decreasing at each iteration. There is no guarantee of success at each iteration, but there is a guarantee of success on average
- To choose the optimal step, we need to know (assume) the number of iterations in advance. Possible solution: initialize T with a small value, after reaching this number of iterations double T and restart the algorithm. A more intelligent way: adaptive selection of stepsize.

Steepest subgradient descent convergence bound

$$\|x_{k+1} - x^*\|^2 = \|x_k - x^* - \alpha_k g_k\|^2 =$$

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$$\frac{1}{T} \left(\sum_{k=0}^{T-1} \langle g_k, x_k - x^* \rangle \right)^2 \leq \sum_{k=0}^{T-1} \langle g_k, x_k - x^* \rangle^2 \leq R^2 G^2 \qquad \sum_{k=0}^{T-1} \langle g_k, x_k - x^* \rangle \leq GR\sqrt{T}$$

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Which leads to exactly the same bound of $\mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$ on the primal gap. In fact, for this class of functions, you can't get a better result than $\frac{1}{\sqrt{T}}$.

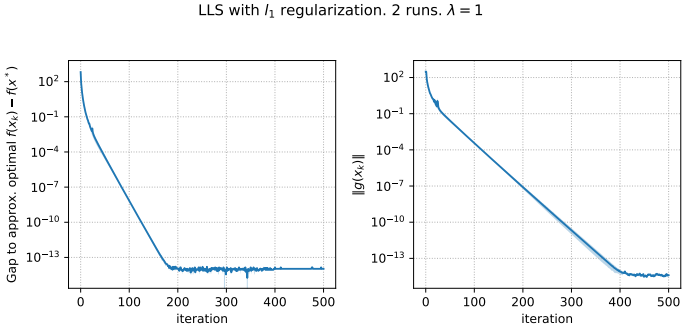
Linear Least Squares with l_1-regularization

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \lambda \|x\|_1$$

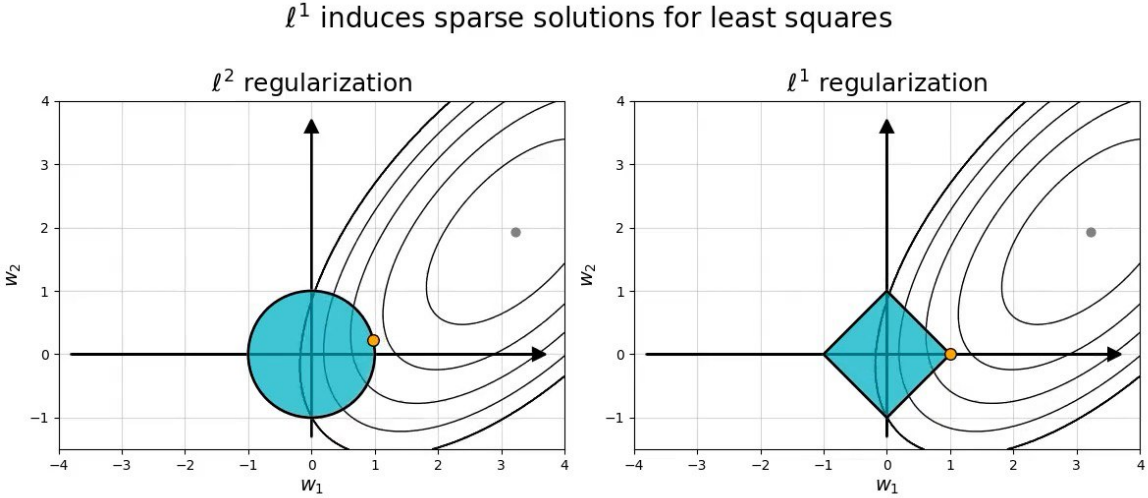
Algorithm will be written as:

$$x_{k+1} = x_k - \alpha_k \left(A^\top (Ax_k - b) + \lambda \mathsf{sign}(x_k) \right)$$

where signum function is taken element-wise.



Great illustration of l_1-regularization



by @itayevron

Support Vector Machines

Let $D = \{(x_i, y_i) \mid x_i \in \mathbb{R}^n, y_i \in \{\pm 1\}\}$

We need to find $\omega \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\min_{\omega \in \mathbb{R}^n, b \in \mathbb{R}} \frac{1}{2}\|\omega\|_2^2 + C \sum_{i=1}^{m} \max[0, 1 - y_i(\omega^\top x_i + b)]$$