1 Definition

An important property of a continuous convex function $f(x)$ is that at any chosen point x_0 for all $x \in \text{dom } f$ the inequality holds:

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$

for some vector g, i.e., the tangent to the graph of the function is the *global* estimate from below for the function.

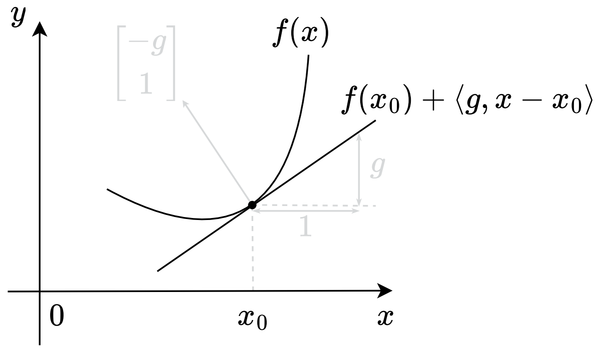


Figure 1: Taylor linear approximation serves as a global lower bound for a convex function

- If $f(x)$ is differentiable, then $g = \nabla f(x_0)$
- Not all continuous convex functions are differentiable 🐱

We wouldn't want to lose such a nice property.

1.1 Subgradient

A vector g is called the **subgradient** of a function $f(x) : S \to \mathbb{R}$ at a point x_0 if $\forall x \in S$:

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$

Example

Find $\partial f(x)$, if $f(x) = |x|$

Solution

The problem can be solved either geometrically (at each point of the numerical line indicate the angular coefficients of the lines globally supporting the function from the bottom), or by the Moreau-Rockafellar theorem, considering $f(x)$ as a point-wise maximum of convex functions:

$$f(x) = \max\{-x, x\}$$

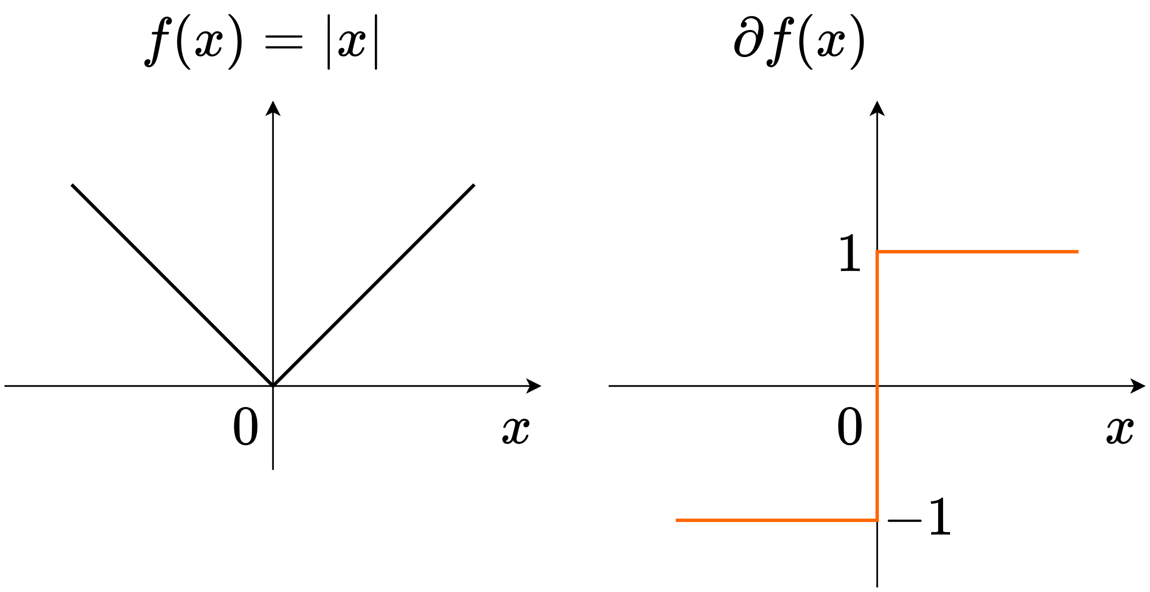


Figure 2: Subgradient of $|x|$

1.2 Subdifferential

The set of all subgradients of a function $f(x)$ at a point x_0 is called the **subdifferential** of f at x_0 and is denoted by $\partial f(x_0)$.

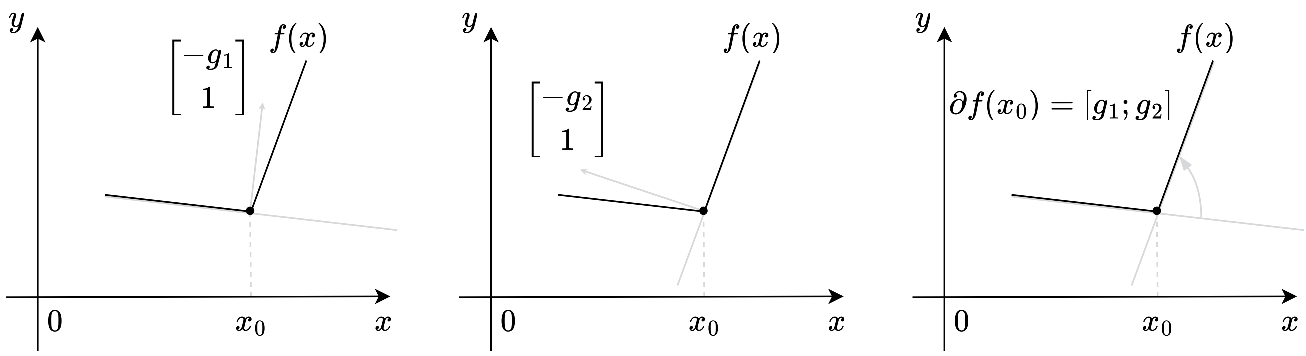


Figure 3: Subgradient calculus

- If $x_0 \in \mathbf{ri} S$, then $\partial f(x_0)$ is a convex compact set.
- The convex function $f(x)$ is differentiable at the point $x_0 \Rightarrow \partial f(x_0) = \{\nabla f(x_0)\}$$
- If $\partial f(x_0) \neq \emptyset \quad \forall x_0 \in S$, then $f(x)$ is convex on S.

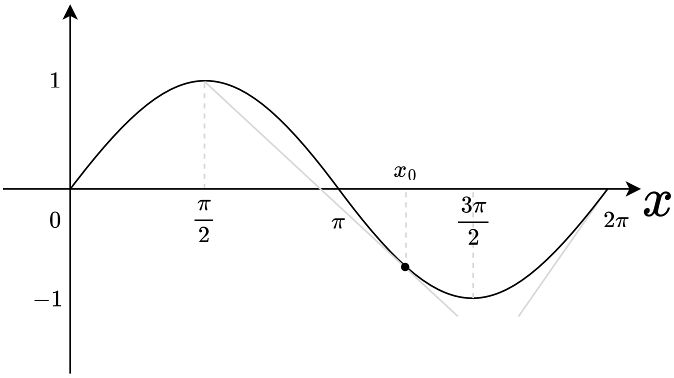
2 Subdifferentiability and convexity

🤔 **Question**

Is it correct, that if the function has a subdifferential at some point, the function is convex?

🧑‍🏫 **Example**

Find $\partial f(x)$, if $f(x) = \sin x, x \in [\pi/2; 2\pi]$



Solution

$$
\partial_S f(x) = \begin{cases} (-\infty; \cos x_0], & x = \frac{\pi}{2} \\ \emptyset, & x \in \left(\frac{\pi}{2}; x_0\right) \\ \cos x, & x \in [x_0; 2\pi) \\ [1; \infty), & x = 2\pi \end{cases}
$$

Theorem

Subdifferential of a differentiable function Let $f : S \to \mathbb{R}$ be a function defined on the set S in a Euclidean space $\mathbb{R}^n$. If $x_0 \in \mathbf{ri}(S)$ and f is differentiable at x_0, then either $\partial f(x_0) = \emptyset$ or $\partial f(x_0) = \{\nabla f(x_0)\}$. Moreover, if the function f is convex, the first scenario is impossible.

Proof

1. Assume, that $s \in \partial f(x_0)$ for some $s \in \mathbb{R}^n$ distinct from $\nabla f(x_0)$. Let $v \in \mathbb{R}^n$ be a unit vector. Because x_0 is an interior point of S, there exists $\delta > 0$ such that $x_0 + tv \in S$ for all $0 < t < \delta$. By the definition of the subgradient, we have

$$
f(x_0 + tv) \geq f(x_0) + t\langle s, v\rangle
$$

which implies:

$$
\frac{f(x_0 + tv) - f(x_0)}{t} \geq \langle s, v\rangle
$$

for all $0 < t < \delta$. Taking the limit as t approaches 0 and using the definition of the gradient, we get:

$$
\langle \nabla f(x_0), v\rangle = \lim_{t \to 0; 0 < t < \delta} \frac{f(x_0 + tv) - f(x_0)}{t} \geq \langle s, v\rangle
$$

2. From this, $\langle s - \nabla f(x_0), v\rangle \geq 0$. Due to the arbitrariness of v, one can set

$$
v = -\frac{s - \nabla f(x_0)}{\|s - \nabla f(x_0)\|},
$$

leading to $s = \nabla f(x_0)$.

3. Furthermore, if the function f is convex, then according to the differential condition of convexity $f(x) \geq f(x_0) + \langle \nabla f(x_0), x - x_0\rangle$ for all $x \in S$. But by definition, this means $\nabla f(x_0) \in \partial f(x_0)$.

It is interesting to mention, that the statement for the convex function could be strengthened. Let $f : S \to \mathbb{R}$ be a convex function defined on the set S in a finite-dimensional Euclidean space $\mathbb{R}^n$, and let $x_0 \in \mathbf{ri}(S)$. Then, f is differentiable at x_0 if and only if the subdifferential $\partial f(x_0)$ contains exactly one element. In this case, $\partial f(x_0) = \{\nabla f(x_0)\}$.

Question

Let $f : S \to \mathbb{R}$ be a function defined on the set S in a Euclidean space, and let $x_0 \in S$. Show that the point x_0 is a minimum of the function f if and only if $0 \in \partial f(x_0)$.

Question

Is it correct, that if the function is convex, it has a subgradient at any point?

Convexity follows from subdifferentiability at any point. A natural question to ask is whether the converse is true: is every convex function subdifferentiable? It turns out that, generally speaking, the answer to this question is negative.

Example

Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) := -\sqrt{x}$. Then, $\partial f(0) = \emptyset$.

Solution

Assume, that $s \in \partial f(0)$ for some $s \in \mathbb{R}$. Then, by definition, we must have $sx \leq -\sqrt{x}$ for all $x \geq 0$. From this, we can deduce $s \leq -\sqrt{1}$ for all $x > 0$. Taking the limit as x approaches 0 from the right, we get $s \leq -\infty$, which is impossible.

3 Subdifferential calculus

Theorem

Moreau - Rockafellar theorem (subdifferential of a linear combination). Пусть $f_i(x)$ - выпуклые функции на выпуклых множествах $S_i, \; i = \overline{1,n}$. Тогда, если $\bigcap\limits_{i=1}^n \mathbf{ri} S_i \neq \emptyset$ то функция $f(x) = \sum\limits_{i=1}^n a_i f_i(x), \; a_i > 0$ имеет субдифференциал $\partial_S f(x)$ на множестве $S = \bigcap\limits_{i=1}^n S_i$ и

$$
\partial_S f(x) = \sum_{i=1}^n a_i \partial_{S_i} f_i(x)
$$

Theorem

Dubovitsky - Milutin theorem (subdifferential of a point-wise maximum). Пусть $f_i(x)$ - выпуклые функции на открытом выпуклом множестве $S \subseteq \mathbb{R}^n, \; x_0 \in S$, а поточечный максимум определяется как $f(x) = \max\limits_i f_i(x)$. Тогда:

$$
\partial_S f(x_0) = \mathbf{conv} \left\{ \bigcup_{i \in I(x_0)} \partial_S f_i(x_0) \right\},
$$

где $I(x) = \{ i \in [1:m] : f_i(x) = f(x) \}$

Chain rule for subdifferentials Пусть $g_1, \ldots, g_m$ - выпуклые функции на открытом выпуклом множестве $S \subseteq \mathbb{R}^n$, $g = (g_1, \ldots, g_m)$ - образованная из них вектор - функция, φ - монотонно неубывающая выпуклая функция на открытом выпуклом множестве $U \subseteq \mathbb{R}^m$, причем $g(S) \subseteq U$. Тогда субдифференциал функции $f(x) = \varphi\left(g(x)\right)$$ имеет вид:

$$
\partial f(x) = \bigcup_{p \in \partial \varphi(u)} \left(\sum_{i=1}^m p_i \partial g_i(x) \right),
$$

где $u = g(x)$

В частности, если функция φ дифференцируема в точке $u = g(x)$, то формула запишется так:

$$
\partial f(x) = \sum_{i=1}^m \dfrac{\partial \varphi}{\partial u_i}(u) \partial g_i(x)
$$

- $\partial (\alpha f)(x) = \alpha \partial f(x)$, for $\alpha \geq 0$
- $\partial (\sum f_i)(x) = \sum \partial f_i(x)$, f_i - выпуклые функции
- $\partial (f(Ax + b))(x) = A^T \partial f(Ax + b)$, f - выпуклая функция
- $z \in \partial f(x)$ if and only if $x \in \partial f^*(z)$.

4 Examples

Концептуально, различают три способа решения задач на поиск субградиента:

- Теоремы Моро - Рокафеллара, композиции, максимума
- Геометрически

- По определению

Example

Найти $\partial f(x)$, если $f(x) = |x-1| + |x+1|$

Solution

Совершенно аналогично применяем теорему Моро - Рокафеллара, учитывая следующее:

$$\partial f_1(x) = \begin{cases} -1, & x < 1 \\ [-1; 1], & x = 1 \\ 1, & x > 1 \end{cases} \qquad \partial f_2(x) = \begin{cases} -1, & x < -1 \\ [-1; 1], & x = -1 \\ 1, & x > -1 \end{cases}$$

Таким образом:

$$\partial f(x) = \begin{cases} -2, & x < -1 \\ [-2; 0], & x = -1 \\ 0, & -1 < x < 1 \\ [0; 2], & x = 1 \\ 2, & x > 1 \end{cases}$$

Example

Найти $\partial f(x)$, если $f(x) = |c_1^\top x| + |c_2^\top x|$

Solution

Пусть $f_1(x) = |c_1^\top x|$, а $f_2(x) = |c_2^\top x|$. Так как эти функции выпуклы, субдифференциал их суммы равен сумме субдифференциалов. Найдем каждый из них:

$$\partial f_1(x) = \partial \left(\max\{c_1^\top x, -c_1^\top x\} \right) = \begin{cases} -c_1, & c_1^\top x < 0 \\ \mathbf{conv}(-c_1; c_1), & c_1^\top x = 0 \\ c_1, & c_1^\top x > 0 \end{cases} \partial f_2(x) = \partial \left(\max\{c_2^\top x, -c_2^\top x\} \right) = \begin{cases} -c_2, & c_2^\top x < 0 \\ \mathbf{conv}(-c_2; c_2), & c_2^\top x = 0 \\ c_2, & c_2^\top x > 0 \end{cases}$$

Далее интересными представляются лишь различные взаимные расположения векторов c_1 и c_2, рассмотрение которых предлагается читателю.

Example

Найти $\partial f(x)$, если $f(x) = [\max(0, f_0(x))]^q$. Здесь $f_0(x)$ - выпуклая функция на открытом выпуклом множестве S, $q \geq 1$.

Solution

Согласно теореме о композиции (функция $\varphi(x) = x^q$ - дифференцируема), а $g(x) = \max(0, f_0(x))$ имеем: $\partial f(x) = q(g(x))^{q-1} \partial g(x)$
По теореме о поточечном максимуме:

$$\partial g(x) = \begin{cases} \partial f_0(x), & f_0(x) > 0, \\ \{0\}, & f_0(x) < 0 \\ \{a \mid a = \lambda a', \ 0 \leq \lambda \leq 1, \ a' \in \partial f_0(x)\}, & f_0(x) = 0 \end{cases}$$

Example

Найти $\partial f(x)$, если $f(x) = \|x\|_1$

Solution

По определению

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n| = s_1 x_1 + s_2 x_2 + \ldots + s_n x_n$$

Рассмотрим эту сумму как поточечный максимум линейных функций по x: $g(x) = s^\top x$, где $s_i = \{-1, 1\}$. Каждая такая функция однозначно определяется набором коэффициентов $\{s_i\}_{i=1}^n$.

Тогда по теореме Дубовицкого - Милютина, в каждой точке $\partial f = \mathbf{conv} \left(\bigcup\limits_{i \in I(x)} \partial g_i(x) \right)$

Заметим, что $\partial g(x) = \partial \left(\max\{s^\top x, -s^\top x\} \right) = \begin{cases} -s, & s^\top x < 0 \\ \mathbf{conv}(-s; s), & s^\top x = 0 \\ s, & s^\top x > 0 \end{cases}$.

Причем, правило выбора “активной” функции поточечного максимума в каждой точке следующее: * Если j-ая координата точки отрицательна, $s_i^j = -1$ * Если j-ая координата точки положительна, $s_i^j = 1$ * Если j-ая координата точки равна нулю, то подходят оба варианта коэффициентов и соответствующих им функций, а значит, необходимо включать субградиенты этих функций в объединение в теореме Дубовицкого - Милютина.

В итоге получаем ответ:

$$\partial f(x) = \{g \ : \ \|g\|_\infty \leq 1, \quad g^\top x = \|x\|_1\}$$

Example

Subdifferential of the Norm. Let V be a finite-dimensional Euclidean space, and $x_0 \in V$. Let $\|\cdot\|$ be an arbitrary norm in V (not necessarily induced by the scalar product), and let $\|\cdot\|_*$ be the corresponding conjugate norm. Then,

$$\partial\|\cdot\|(x_0) = \begin{cases} B_{\|\cdot\|_*}(0,1), & \text{if } x_0 = 0, \\ \{s \in V : \|s\|_* \leq 1; \langle s, x_0 \rangle = \|x_0\|\} = \{s \in V : \|s\|_* = 1; \langle s, x_0 \rangle = \|x_0\|\}, & \text{otherwise.} \end{cases}$$

Where $B_{\|\cdot\|_*}(0,1)$ is the closed unit ball centered at zero with respect to the conjugate norm. In other words, a vector $s \in V$ with $\|s\|_* = 1$ is a subgradient of the norm $\|\cdot\|$ at point $x_0 \neq 0$ if and only if the Hölder's inequality $\langle s, x_0 \rangle \leq \|x_0\|$ becomes an equality.

Proof

Let $s \in V$. By definition, $s \in \partial\|\cdot\|(x_0)$ if and only if

$$\langle s, x \rangle - \|x\| \leq \langle s, x_0 \rangle - \|x_0\|, \text{ for all } x \in V,$$

or equivalently,

$$\sup_{x \in V}\{\langle s, x \rangle - \|x\|\} \leq \langle s, x_0 \rangle - \|x_0\|.$$

By the definition of the supremum, the latter is equivalent to

$$\sup_{x \in V}\{\langle s, x \rangle - \|x\|\} = \langle s, x_0 \rangle - \|x_0\|.$$

It is important to note that the expression on the left side is the supremum from the definition of the Fenchel conjugate function for the norm, which is known to be

$$\sup_{x \in V}\{\langle s, x \rangle - \|x\|\} = \begin{cases} 0, & \text{if } \|s\|_* \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, equation is equivalent to $\|s\|_* \leq 1$ and $\langle s, x_0 \rangle = \|x_0\|$.

Consequently, it remains to note that for $x_0 \neq 0$, the inequality $\|s\|_* \leq 1$ must become an equality since, when $\|s\|_* < 1$, Hölder's inequality implies $\langle s, x_0 \rangle \leq \|s\|_* \|x_0\| < \|x_0\|$.

The conjugate norm in Example above does not appear by chance. It turns out that, in a completely similar manner for an arbitrary function f (not just for the norm), its subdifferential can be described in terms of the dual object — the Fenchel conjugate function.

Example

Characterization of the subdifferential through the conjugate function. Let $f : E \to \mathbb{R}$ be a function defined on the set E in a Euclidean space. Let $x_0 \in E$ and let $f^* : E^* \to \mathbb{R}$ be the conjugate function. Show that

$$\partial f(x_0) = \{s \in E^* : \langle s, x_0 \rangle = f^*(s) + f(x_0)\},$$

In other words, a vector $s \in E^*$ is a subgradient of the function f at point x_0 if and only if the Fenchel-Young inequality $\langle s, x_0 \rangle \leq f^*(s) + f(x_0)$ becomes an equality.

In the case $f = \|\cdot\|$, we have $f^* = \delta_{B_{\|\cdot\|_*}(0,1)}$, i.e., the conjugate function is equal to the indicator function of the ball $B_{\|\cdot\|_*}(0,1)$, and equation becomes.

Theorem

Criteria for equality in the Fenchel-Young inequality. Let $f : E \to \mathbb{R}$ be a convex closed function, $f^* : E^* \to \mathbb{R}$ the conjugate function, and let $x \in E, s \in E^*$. The following statements are equivalent:

a. $\langle s, x \rangle = f^*(s) + f(x)$.
b. $s \in \partial f(x)$.
c. $x \in \partial f^*(s)$.

Proof

According to Exercise above, the condition $\langle s, x \rangle = f^*(s) + f(x)$ is equivalent to $s \in \partial f(x)$. On the other hand, since f is convex and closed, by the Fenchel-Moreau theorem, we have $f^{**} = f$. Applying Exercise 1.13 to the function f^*, it follows that the equality $\langle s, x \rangle = f^*(s) + f(x)$ is equivalent to $x \in \partial f^*(s)$.

5 References

- Lecture Notes for ORIE 6300: Mathematical Programming I by Damek Davis